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EXISTENCE THEOREMS FOR TWO-DIMENSIONAL LINEAR SHELL THEORIES

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EXISTENCE THEOREMS FOR TWO-DIMENSIONAL LINEAR SHELL THEORIES[1 2]

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Abstract

We consider linearly elastic shells whose middle surfaces have the most general geometries, and we provide complete proofs of the ellipticity of the strain energies found in two commonly used two-dimensional models : Koiter's model and Naghdi's model.

THEOREMES D'EXISTENCE POUR DES THEORIES BIDIMENSIONNELLES LINEAIRES DE COQUES

Résumé

Nous considérons des coques élastiques linéaires dont les surfaces moyennes ont les formes les plus générales possibles, et nous donnons les démonstrations complètes de l'ellipticité des énergies de déformation des deux modèles les plus couramment utilisés : le modèle de Koiter et le modèle de Naghdi.

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Contents

INTRODUCTION

Establishing *existence theorems* for *two-dimensional linear shell theories* poses considerable more difficulties than for plates, the added difficulties stemming from the "geometry" of the middle surface of the shell under consideration. In each case, the key step of the proof consists in establishing the *ellipticity* of the associated bilinear form, also called the *strain energy*, over an appropriate functional space ; then the existence and uniqueness of the solution follow from the *Lax-Milgram lemma*.

In addition to its mathematical interest *per se*, such an existence result in the linear case is an indispensable tool for establishing *existence theorems for two-dimensional nonlinear shell theories* via the implicit function theorem, and *error estimates for the numerical approximation of the solution*, usually by finite elements (cf. Bernadou & Boisserie [1982], Ciarlet [1978]).

In this paper, we consider shells whose middle surfaces have the *most general "geometries"* and we provide complete proofs of the ellipticity of the strain energies associated with two commonly used models in linear two-dimensional shell theories : *Koiter's model*, and *Naghdi's model*. We do not discuss here the *validity* of these models, nor their relative merits ; we simply indicate in each instance according to which guidelines the model under consideration can be derived.

After a brief review in Sect. 1 of those notions of *differential geometry* that will be subsequently needed, we first consider in Sect. 2 *Koiter's model*, named after Koiter [1966,1970]. The ellipticity of its associated strain energy was first established in Bernadou & Ciarlet [1976], by means of a proof that was substantially "technical". We use here a different, and simpler approach, first announced in Ciarlet & Miara [1991] (see also Ciarlet & Miara [1992a]).

The main novelty consists in using a crucial *lemma of J.L. Lions*, which allows for considerable simplification and "transparency" ; in particular, the proof is now more reminiscent, apart from all the technicalities due to the "geometry" of the middle surface of the shell, of the proof of Korn's inequality given in Duvaut & Lions [1972] (and which also relies on the same lemma of J.L. Lions). Note however that we retain one essential step from the former proof of Bernadou & Ciarlet [1976], namely the *rigid displacement lemma* (cf. Lemma 2.5).

To our knowledge, the first instance where the same lemma of J.L. Lions was used in connection with a "genuine" shell theory (i.e., neither in three-dimensional elasticity nor in two-dimensional plate theory) was for establishing a *generalized Korn's inequality* that plays a key rôle in the proof that the three-dimensional solution of a specific class of *shallow shell problems* converges to the solution of a two-dimensional problem (cf. Ciarlet & Miara

[1992b]) as the thickness of the shell approaches zero.

While Koiter's model belongs to the family of *Kirchhoff-Love theories*, thus named after Kirchhoff [1876] and Love [1934], different mathematical models for shells that rely on the theory of *Cosserat surfaces* (cf. Cosserat & Cosserat [1909]) have been proposed by Naghdi [1963,1972]. We then show in Sect. 3 that a similar use of the lemma of J.L. Lions, together with another *rigid displacement lemma*, due to Coutris [1978], again yields the proof of the ellipticity of the corresponding strain energy (this approach was also announced in Ciarlet & Miara [1991]).

Various other strain energies have been proposed for modeling linearly elastic shells, and their ellipticity has been accordingly studied by various authors. In this respect, we notably mention Rougée [1969] for cylindrical shells, Gordeziani [1974] for the shell model proposed by Vekua [1965], Shoikhet [1974] for the shell model proposed by Novozhilov [1970], Bernadou & Lalanne [1985] for the shallow shell model proposed by Koiter [1966, eqs. (11.43) and (11.44)], Ciarlet & Miara [1992b] for the linearized Marguerre-von Kármán shallow shell model.

A very interesting "limit analysis" of Koiter's model when the thickness approaches zero has been recently given by Sanchez-Palencia [1989a,1989b]. It in turn serves as a basis for a new approach for the derivation of two-dimensional shell models from three-dimensional elasticity (cf. Sanchez-Palencia [1990]), which significantly complements the pioneering work of Destuynder [1980,1985] ; see also Ciarlet [1992a,1992b].

Two essentially different limit models are then found, according to the geometry of the middle surface of the shell and the imposed boundary conditions : either a *bending-dominated model*, or a *membrane-dominated model*. While the proof of the ellipticity of the *bending-dominated model* is a simple corollary of that of Koiter's model, proving the ellipticity of the *membrane-dominated model* poses considerable difficulties ; in this direction, see Geymonat & Sanchez-Palencia [1991] in the "uniformly elliptic" case, and Piila & Pitkäranta [1992] in the "parabolic" case.

It is worth noticing that, by contrast, *no such restrictions on the geometry of the middle surface* (we only assume that the mapping that defines it is smooth enough) *or on the boundary conditions* (we only assume that they are imposed on a set of strictly positive length) *need be imposed in the two models that we are considering here.*

1 GEOMETRY OF THE MIDDLE SURFACE OF THE SHELL

Among the many references about the differential geometry of surfaces, those of Valiron [1950], Green & Zerna [1968], Naghdi [1972] and Bernadou & Boisserie [1982], are particularly appropriate for our purposes.

In what follows, Greek indices and exponents always belong to the set $\{1,2\}$, while Latin indices and exponents (unless they are used to index a sequence) always belong to the set $\{1,2,3\}$; the summation convention is also systematically used ; finally we assume that an origin and an orthonormal basis $(\mathbf{e}_i)$ are given once and for all in the Euclidean space, which will be accordingly identified with $\mathcal{R}^3$.

Let ω denote a two-dimensional bounded open connected set ; we assume that its boundary γ is Lipschitz-continuous in the sense of Nečas [1967]. Let (x^1, x^2) denote a generic point in the set $\bar{\omega}$, and let $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$, $\partial_{\alpha\beta} = \frac{\partial^2}{\partial x^\alpha \partial x^\beta}$.

We shall consider a *shell*, whose *middle surface* S is the image of the set $\bar{\omega}$ through a mapping $\varphi = \varphi^i \mathbf{e}_i : \bar{\omega} \to \mathcal{R}^3$; we assume once and for all that the mapping φ is injective, of class $\mathcal{C}^3$, and that the two vectors

$$\mathbf{a}_\alpha = \partial_\alpha \varphi = (\partial_\alpha \varphi^i)\mathbf{e}_i \tag{1.1}$$

are linearly independent at each point $(x^1, x^2) \in \bar{\omega}$. This being the case, the two vectors $\mathbf{a}_\alpha(x^1, x^2)$, which are tangent to the *coordinate curves* passing through the point $\varphi(x^1, x^2)$ (the coordinates curves are the images through φ of the portions of the lines "x^α = constant" that are contained in the set $\bar{\omega}$), span the *tangent plane* to the surface $S = \varphi(\bar{\omega})$ at the point $\varphi(x^1, x^2)$ (cf. Figure 1.1) : they form the *covariant basis of the tangent plane* at that point.

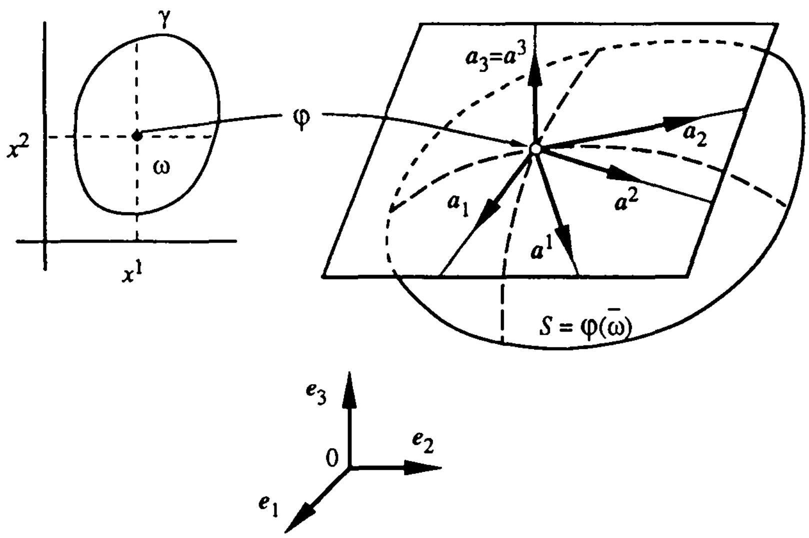

Figure 1.1 : At each point of the middle surface $S = \varphi(\bar{\omega})$ of the shell, the vectors $\mathbf{a}_\alpha = \partial_\alpha \varphi$, $\alpha = 1, 2$, form the covariant basis of the tangent plane, the vectors $\mathbf{a}^\alpha$ defined by $\mathbf{a}^\alpha . \mathbf{a}_\beta = \delta^\alpha_\beta$ form the contravariant basis of the tangent plane and the vector $\mathbf{a}^3 = \mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}$ is normal to the tangent plane. The vectors $\mathbf{a}_i$, $i = 1, 2, 3$, form the covariant basis, and the vectors $\mathbf{a}^i$ form the contravariant basis, at the same point.

Remarks : (1) The assumption that φ is of class $\mathcal{C}^3$ is needed to insure that the functions $\partial_\alpha b^\rho_\beta$ appearing in (2.7) are continuous on $\bar{\omega}$.

(2) The tangent plane to S at $\varphi(x^1, x^2)$ is in fact the affine plane that passes through $\varphi(x^1, x^2)$ and that is parallel to the two vectors $\mathbf{a}_\alpha(x^1, x^2)$, i.e., it is a *translated* plane. To avoid cumbersome statements, we shall not however make explicit mention, here

or subsequently, of such translations. ■

Let $\mathbf{u}.\mathbf{v}$ denote the Euclidean inner product of two vectors $\mathbf{u} \in \mathcal{R}^3$ and $\mathbf{v} \in \mathcal{R}^3$, let $|.|$ denote the associated Euclidean norm, and let $\mathbf{u} \times \mathbf{v}$ denote the vector product of $\mathbf{u}$ and $\mathbf{v}$. At each point $(x^1, x^2) \in \bar{\omega}$, we define the vector

$$\mathbf{a}_3 = \mathbf{a}^3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}. \tag{1.2}$$

The vectors $\mathbf{a}_\alpha$ defined in (1.1) and the vector $\mathbf{a}_3$ thus define a basis at the point $\varphi(x^1, x^2)$, which is called the *covariant basis* at that point (cf. Figure 1.1).

We also define at each point $\varphi(x^1, x^2)$ two vectors $\mathbf{a}^\alpha$ of the tangent plane by the relations

$$\mathbf{a}^\alpha.\mathbf{a}_\beta = \delta^\alpha_\beta, \tag{1.3}$$

where δ^α_β, and δ^i_j in (1.4) below, denote the Kronecker symbol. The vectors $\mathbf{a}^\alpha$ defined in (1.3) form the *contravariant basis of the tangent plane* and the vectors $\mathbf{a}^i$ defined in (1.2)-(1.3) form the *contravariant basis*, at the point considered. Note that they satisfy

$$\mathbf{a}^i.\mathbf{a}_j = \delta^i_j, \tag{1.4}$$

$$\mathbf{a}_\alpha \times \mathbf{a}_\beta = \varepsilon_{\alpha\beta}\mathbf{a}^3, \quad \mathbf{a}^\alpha \times \mathbf{a}^\beta = \varepsilon^{\alpha\beta}\mathbf{a}_3, \quad \mathbf{a}_3 \times \mathbf{a}_\beta = \varepsilon_{\beta\rho}\mathbf{a}^\rho, \quad \mathbf{a}^3 \times \mathbf{a}^\beta = \varepsilon^{\beta\rho}\underset{\sim}{a}_\rho, \tag{1.5}$$

where (a is defined in (1.8)) :

$$(\varepsilon_{\alpha\beta}) = \sqrt{a}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (\varepsilon^{\alpha\beta}) = \frac{1}{\sqrt{a}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.6}$$

The *first fundamental form*, or *metric tensor*, $(a_{\alpha\beta})$ of the surface S is defined by

$$a_{\alpha\beta} = a_{\beta\alpha} = \mathbf{a}_\alpha.\mathbf{a}_\beta = \partial_\alpha\varphi^i\partial_\beta\varphi^i \tag{1.7}$$

(it is used for evaluating length of curves on S). Note that by assumption, the determinant

$$a = det(a_{\alpha\beta}) \tag{1.8}$$

is > 0 in $\bar{\omega}$; hence there exists a constant a_0 such that

$$a(x^1, x^2) \geq a_0 > 0 \text{ for all } (x^1, x^2) \in \bar{\omega}. \tag{1.9}$$

The area element dS along S is then given by

$$dS = \sqrt{a}dx^1dx^2. \tag{1.10}$$

We also let

$$a^{\alpha\beta} = \mathbf{a}^\alpha.\mathbf{a}^\beta, \tag{1.11}$$

so that the matrix $(a^{\alpha\beta})$ is the inverse of the matrix $(a_{\alpha\beta})$ defined in (1.7).

The *second fundamental form* $(b_{\alpha\beta})$ of the surface S is defined by

$$b_{\alpha\beta} = b_{\beta\alpha} = \mathbf{a}_3.\partial_\alpha\mathbf{a}_\beta = -\mathbf{a}_\alpha.\partial_\alpha\mathbf{a}_\beta \tag{1.12}$$

(it is used for measuring the curvature of the surface). We also let

$$b_\alpha^\beta = a^{\beta\rho} b_{\rho\alpha}. \tag{1.13}$$

The *third fundamental form* $(c_{\alpha\beta})$ of the surface S is defined by

$$c_{\alpha\beta} = c_{\beta\alpha} = b_\alpha^\rho b_{\rho\beta} = b_{\alpha\rho} b_\beta^\rho. \tag{1.14}$$

The *Christoffel symbols* $\Gamma_{\alpha\beta}^\rho$ of the surface, which are defined by

$$\Gamma_{\alpha\beta}^\rho = \Gamma_{\beta\alpha}^\rho = \mathbf{a}^\rho . \partial_\beta \mathbf{a}_\alpha, \tag{1.15}$$

are used for computing *covariant derivatives* : For instance, let

$$\eta_\alpha \mathbf{a}^\alpha = \eta^\alpha \mathbf{a}_\alpha$$

be a *surface vector field* ; then the functions

$$\eta_{\alpha|\beta} = \partial_\beta \eta_\alpha - \Gamma_{\alpha\beta}^\rho \eta_\rho, \quad \eta^\alpha|_\beta = \partial_\beta \eta^\alpha + \Gamma_{\rho\beta}^\alpha \eta^\rho, \tag{1.16}$$

are the *covariant derivatives of the vector field.* Likewise, let $T_{\alpha\beta}$ and $T^{\alpha\beta}$ denote the covariant and contravariant components of a *surface tensor field* ; then the functions

$$T_{\alpha\beta|\rho} = \partial_\rho T_{\alpha\beta} - \Gamma_{\alpha\rho}^\sigma T_{\sigma\beta} - \Gamma_{\beta\rho}^\sigma T_{\alpha\sigma}, \quad T^{\alpha\beta}|_\rho = \partial_\rho T^{\alpha\beta} + \Gamma_{\sigma\rho}^\alpha T^{\sigma\beta} + \Gamma_{\sigma\rho}^\beta T^{\alpha\sigma}, \tag{1.17}$$

are the *covariant derivatives of the tensor field.* Note that, in particular

$$\varepsilon^{\alpha\beta}|_\rho = 0, \tag{1.18}$$

where $(\varepsilon^{\alpha\beta})$ is the tensor defined in (1.6).

If $T_{\alpha\beta}$ are the covariant components of a *symmetric* surface tensor, its *mixed components* $T_\alpha^\beta = a^{\beta\rho} T_{\alpha\rho}$ are unambiguously defined, and their covariant derivatives are given by

$$T_\alpha^\beta|_\rho = \partial_\rho T_\alpha^\beta + \Gamma_{\rho\sigma}^\beta T_\alpha^\sigma - \Gamma_{\alpha\rho}^\sigma T_\sigma^\beta. \tag{1.19}$$

The covariant derivatives of the mixed tensor (b_α^β) defined in (1.13) satisfy in addition the following *symmetry relations*

$$b_\alpha^\beta|_\rho = b_\rho^\beta|_\alpha, \tag{1.20}$$

which themselves follow from the *Mainardi-Codazzi identities*

$$b_{\alpha\beta|\rho} = b_{\alpha\rho|\beta}. \tag{1.21}$$

Let next

$$\eta_i \mathbf{a}^i = \eta^i \mathbf{a}_i$$

be a vector field defined along the surface S ; then its partial derivatives are given by (recall that $\mathbf{a}_3 = \mathbf{a}^3$, and note that $\eta_3 = \eta^3$) :

$$\left\{ \begin{aligned} \partial_\alpha(\eta_i \mathbf{a}^i) &= (\eta_{\beta|\alpha} - b_{\alpha\beta}\eta_3)\mathbf{a}^\beta + (\partial_\alpha \eta_3 + b_\alpha^\beta \eta_\beta)\mathbf{a}^3 \\ &= (\eta^\beta|_\alpha - b_\alpha^\beta \eta_3)\mathbf{a}_\beta + (\partial_\alpha \eta^3 + b_{\alpha\beta}\eta^\beta)\mathbf{a}_3. \end{aligned} \right. \tag{1.22}$$

Relations (1.22) themselves follow from the *formulas of Gauss* :

$$(1.23) \qquad \partial_\beta \mathbf{a}_\alpha = \Gamma^\rho_{\alpha\beta}\mathbf{a}_\rho + b_{\alpha\beta}\mathbf{a}_3, \quad \partial_\beta \mathbf{a}^\alpha = -\Gamma^\alpha_{\beta\rho}\mathbf{a}^\rho + b^\alpha_\beta \mathbf{a}_3,$$

and Weingarten :

$$(1.24) \qquad \partial_\alpha \mathbf{a}_3 = -b^\rho_\alpha \mathbf{a}_\rho.$$

Note that

$$(1.25) \qquad (T^{\alpha\rho}\eta_\rho)|_\beta = (T^{\alpha\rho}|_\beta)\eta_\rho + T^{\alpha\rho}(\eta_{\rho|\beta}), \quad (T^\rho_\alpha\eta_\rho)|_\beta = (T^\rho_\alpha|_\beta)\eta_\rho + T^\rho_\alpha(\eta_{\rho|\beta}),$$

i.e., covariant derivatives are computed in these cases according to the same rule as if they were "usual" derivatives (we assume here that $(T_{\alpha\beta})$ is a symmetric tensor, so that we can use (1.19)).

Since the covariant derivatives $\eta_{\alpha|\beta}$ are themselves the components of surface tensors, their covariant derivatives

$$(1.26) \qquad \eta_{\rho|\alpha\beta} := (\eta_{\rho|\alpha})|_\beta$$

can themselves be computed according to formulas (1.17). The functions $\eta_{\alpha|\beta\rho}$ as defined in (1.26) are called the *second covariant derivatives* of the vector field $\eta_\alpha \mathbf{a}^\alpha$. Note in this respect that the second covariant derivatives $\eta_{\rho|\alpha\beta}$ and $\eta_{\rho|\beta\alpha}$ do *not* commute in general ; instead, one has

$$(1.27) \qquad \eta_{\rho|\alpha\beta} - \eta_{\rho|\beta\alpha} = b_{\rho\beta}b^\sigma_\alpha\eta_\sigma - b_{\rho\alpha}b^\sigma_\beta\eta_\sigma.$$

Likewise, since the functions

$$(1.28) \qquad \eta_{3|\alpha} = \eta^3|_\alpha = \partial_\alpha\eta_3$$

are the components of a surface tensor, their covariant derivatives are given by

$$(1.29) \qquad \eta_{3|\alpha\beta} := (\eta_{3|\alpha})|_\beta = \partial_{\alpha\beta}\eta_3 - \Gamma^\rho_{\alpha\beta}\partial_\rho\eta_3 = \eta_{3|\beta\alpha}.$$

2 EXISTENCE THEORY FOR KOITER'S MODEL

Let the surface $S = \varphi(\bar\omega)$ be defined, and let the notation be the same, as in Sect. 1. A *shell*, with *middle surface* S, and with *thickness* $2\varepsilon > 0$, is an elastic body whose *reference configuration* is the closure of the set :

$$\hat\Omega = \{(\varphi(x^1,x^2) + x^3\mathbf{a}_3(x^1,x^2)) \in \mathcal{R}^3 \ ; \ (x^1,x^2) \in \omega, \quad |x^3| < \varepsilon\}.$$

In order to be in a physically realistic situation, we assume of course that the mapping $\boldsymbol{\Phi} : (\bar\omega \times [-\varepsilon,\varepsilon]) \to \mathcal{R}^3$ defined by $\boldsymbol{\Phi}(x^1,x^2,x^3) = \varphi(x^1,x^2) + x^3\mathbf{a}_3(x^1,x^2)$ for $(x^1,x^2,x^3) \in (\bar\omega \times [-\varepsilon,\varepsilon])$ is injective. Since the mapping $\varphi : \bar\omega \to \mathcal{R}^3$ is smooth and injective by assumption, this is the case if $\varepsilon > 0$ is small enough (cf. Ciarlet & Paumier [1986]).

For simplicity, we restrict our attention here to linearly elastic materials that are *homogeneous* and *isotropic* (and of course whose reference configuration is a natural state, so that linearized elasticity is a meaningful approximation ; cf. e.g. Ciarlet [1988, sect. 6.2]), but it should be noted that the extension to nonhomogeneous or to anisotropic materials poses no difficulties other than technical.

We assume that the *Lamé constants* λ and μ of the material that constitutes the shell satisfy

$$\lambda > 0, \quad \mu > 0, \tag{2.1}$$

these inequalities being suggested by physical evidence for actual elastic materials. The shell is subjected to applied body forces in its interior $\hat{\Omega}$ and to applied surface forces on its upper and lower faces

$$\hat{\Gamma}_{\pm} = \{(\varphi(x^1,x^2) + x^3\mathbf{a}_3(x^1,x^2)) \in \mathcal{R}^3 \; ; \; (x^1,x^2) \in \omega, \quad x^3 = \pm\varepsilon\},$$

and it is *clamped* on a portion Γ_0 of its lateral surface, of the form :

$$\hat{\Gamma}_0 = \{(\varphi(x^1,x^2) + x^3\mathbf{a}_3(x^1,x^2)) \in \mathcal{R}^3 \; ; \; (x^1,x^2) \in \gamma_0, \quad |x^3| \leq \varepsilon\},$$

where γ_0 denotes a measurable subset of the boundary of ω (this means that the unknown displacement vector field is required to vanish on $\hat{\Gamma}_0$). Finally, the shell may be also subjected to surface forces along the remaining portion

$$\hat{\Gamma}_1 = \{(\varphi(x^1,x^2) + x^3\mathbf{a}_3(x^1,x^2)) \in \mathcal{R}^3 \; ; \; (x^1,x^2) \in \gamma_1, \quad |x^3| \leq \varepsilon\},$$

where $\gamma_1 := \gamma - \gamma_0$, of its lateral surface.

For each integer $m \geq 0$, we let $H^m(\omega)$ and $\|.\|_{m,\omega}$ denote the usual Sobolev space and norm ; in particular, if η is a real-valued function defined over ω, we have

$$\|\eta\|_{0,\omega} = \left\{\int_\omega |\eta|^2 d\omega\right\}^{1/2}, \quad \|\eta\|_{1,\omega} = \left\{\|\eta\|_{0,\omega}^2 + \sum_\alpha \|\partial_\alpha \eta\|_{0,\omega}^2\right\}^{1/2},$$

$$\|\eta\|_{2,\omega} = \left\{\|\eta\|_{1,\omega}^2 + \sum_{\alpha,\beta} \|\partial_{\alpha\beta}\eta\|_{0,\omega}^2\right\}^{1/2}.$$

We use boldface letters for denoting vector-valued fonctions and their associated function spaces.

In his pioneering work on shells, John [1965,1971] has shown that, if the thickness is small enough, the state of stress inside the shell is "approximately" planar, and that the stresses parallel to the middle surface vary "approximately linearly" across the thickness. In *Koiter's approach* (cf. Koiter [1966,1970]), these approximations are taken as *a priori* assumptions and combined with another *a priori* assumption, of a geometrical nature (cf. Koiter [1966, pp. 15-16]) : any point that is situated on a normal to the middle surface remains, after the deformation has taken place, on the normal to the deformed middle surface ; this is the first part of the *Kirchhoff-Love assumption*, the second part asserting that, in addition, the distance between such a point and the middle surface remains constant.

Taking these *a priori* assumptions into account, W.T. Koiter then shows that the displacement field across the thickness of the shell can be completely determined from the sole knowledge of the displacement field of the points of the *middle surface* S, and he identifies the *two-dimensional problem*, i.e., posed over the two-dimensional set $\bar{\omega}$, that this displacement field should solve.

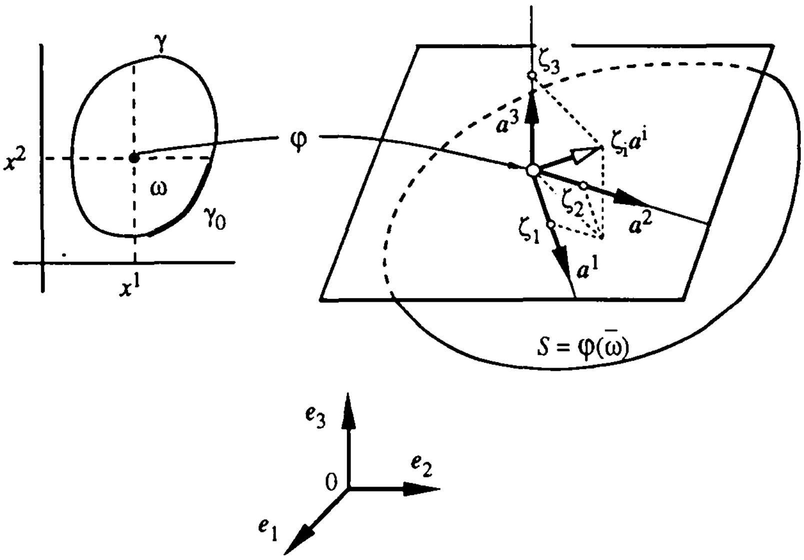

Figure 2.1 : In Koiter's model, the three unknowns are the covariant components $\zeta_i : \bar{\omega} \to \mathcal{R}$ of the displacement $\zeta_i \mathbf{a}^i$ of the points of the middle surface $S = \varphi(\bar{\omega})$ of the shell. The vectors $\mathbf{a}^\alpha$ form the contravariant basis of the tangent plane, and $\mathbf{a}^3 = \frac{\mathbf{a}^1 \times \mathbf{a}^2}{|\mathbf{a}^1 \times \mathbf{a}^2|}$; cf. Figure 1.1.

More specifically, let $\zeta_i : \bar{\omega} \to \mathcal{R}$ denote the three *covariant components* of the *displacement $\zeta_i \mathbf{a}^i$ of the points of the middle surface S*. This means that $\zeta_i(x^1, x^2)\mathbf{a}^i(x^1, x^2)$ is the displacement of the point $\varphi(x^1, x^2)$ for all points $(x^1, x^2) \in \bar{\omega}$; cf. Figure 2.1. Then *the unknown*

$$\boldsymbol{\zeta} = (\zeta_i)$$

solves the following variational problem, called **Koiter's shell model** :

$$\text{(2.2)} \qquad \boldsymbol{\zeta} \in \mathbf{V}(\omega) \text{ and } B(\boldsymbol{\zeta}, \boldsymbol{\eta}) = L(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \mathbf{V}(\omega),$$

where the space $\mathbf{V}(\omega)$ is defined as (∂_ν denotes the outer normal derivative operator along γ) :

$$\text{(2.3)} \qquad \mathbf{V}(\omega) = \{\boldsymbol{\eta} = ((\eta_\alpha), \eta_3) \in \mathbf{H}^1(\omega) \times H^2(\omega) \, ; \, \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

where the symmetric bilinear form B is defined by :

$$\text{(2.4)} \qquad B(\boldsymbol{\zeta}, \boldsymbol{\eta}) = \int_\omega \left\{ \varepsilon a^{\alpha\beta\rho\sigma} \gamma_{\rho\sigma}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} \, a^{\alpha\beta\rho\sigma} \Upsilon_{\rho\sigma}(\boldsymbol{\zeta}) \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) \right\} \sqrt{a} d\omega,$$

with (cf. (1.11)-(1.15) for the definitions of the functions $a^{\alpha\beta}$, $b_{\alpha\beta}$, b_α^β, $c_{\alpha\beta}$, $\Gamma^\rho_{\alpha\beta}$ and (1.16), (1.19), (1.29) for the definitions of the covariant derivatives) :

$$\text{(2.5)} \qquad a^{\alpha\beta\rho\sigma} = \frac{4\lambda\mu}{(\lambda + 2\mu)} \, a^{\alpha\beta} a^{\rho\sigma} + 2\mu(a^{\alpha\rho} a^{\beta\sigma} + a^{\alpha\sigma} a^{\beta\rho}),$$

$$
(2.6) \quad \begin{cases} \gamma_{\alpha\beta}(\boldsymbol{\eta}) & = \frac{1}{2}\,(\eta_{\alpha|\beta} + \eta_{\beta|\alpha}) - b_{\alpha\beta}\eta_3 \\ & = e_{\alpha\beta}(\boldsymbol{\eta}) - \Gamma^{\rho}_{\alpha\beta}\eta_\rho - b_{\alpha\beta}\eta_3, \text{ with } e_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}\,(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha), \end{cases}
$$

$$
(2.7) \quad \begin{cases} \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) & = \eta_{3|\alpha\beta} + b^{\rho}_{\beta}\eta_{\rho|\alpha} + b^{\rho}_{\alpha}\eta_{\rho|\beta} + (b^{\rho}_{\beta}|_\alpha)\eta_\rho - c_{\alpha\beta}\eta_3 \\ & = \partial_{\alpha\beta}\eta_3 - \Gamma^{\rho}_{\alpha\beta}\partial_\rho\eta_3 + b^{\rho}_{\beta}(\partial_\alpha\eta_\rho - \Gamma^{\sigma}_{\rho\alpha}\eta_\sigma) + b^{\rho}_{\alpha}(\partial_\beta\eta_\rho - \Gamma^{\sigma}_{\rho\beta}\eta_\sigma) \\ & \quad + (\partial_\alpha b^{\rho}_{\beta} + \Gamma^{\rho}_{\alpha\sigma}b^{\sigma}_{\beta} - \Gamma^{\sigma}_{\alpha\beta}b^{\rho}_{\sigma})\eta_\rho - c_{\alpha\beta}\eta_3, \end{cases}
$$

and where L is the linear form that takes into account the applied forces. The functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\Upsilon_{\alpha\beta}(\boldsymbol{\eta})$ are the covariant components of the *linearized strain*, and *change of curvature, tensors* associated with an arbitrary displacement field $\boldsymbol{\eta}$ of the surface S. Note that both tensors are *symmetric*, i.e.,

$$
(2.8) \qquad \gamma_{\alpha\beta}(\boldsymbol{\eta}) = \gamma_{\beta\alpha}(\boldsymbol{\eta}) \text{ and } \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) = \Upsilon_{\beta\alpha}(\boldsymbol{\eta}),
$$

the second equalities in (2.8) being in particular a consequence of the symmetry relations (1.20). We refer to (1.8) and (1.10) for the meaning of $\sqrt{a}$.

Remark : The linear form L takes the form

$$
L(\boldsymbol{\eta}) = \int_\omega \mathbf{p}.\boldsymbol{\eta}\sqrt{a}d\omega + \int_{\gamma_1} \mathbf{q}.\boldsymbol{\eta}d\gamma + \int_{\gamma_1} m^\alpha(\partial_\alpha\eta_3 + b^{\sigma}_{\alpha}\eta_\sigma)d\gamma, \text{ for all } \boldsymbol{\eta} \in \mathbf{V}(\omega),
$$

where $\gamma_1 = \gamma - \gamma_0$ and where the vector fields $\mathbf{p} : \omega \to \mathcal{R}^3$, $\mathbf{q} : \gamma_1 \to \mathcal{R}^3$ and $(m^\alpha) : \gamma_1 \to \mathcal{R}^2$ are determined, through appropriate integration across the thickness of the shell, from the knowledge of the given applied body forces in $\hat{\Omega}$ and applied surface forces on $\hat{\Gamma}_+ \cup \hat{\Gamma}_- \cup \hat{\Gamma}_1$. ■

Let

$$
(2.9) \qquad \|\boldsymbol{\eta}\|_{\mathbf{H}^1(\omega)\times H^2(\omega)} = \left\{\sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega}\right\}^{1/2}
$$

denote the norm of an element $\boldsymbol{\eta} = ((\eta_\alpha), \eta_3) \in \mathbf{V}(\omega)$, where $\mathbf{V}(\omega)$ is the closed subspace of $\mathbf{H}^1(\omega) \times H^2(\omega)$ defined in (2.3). We now establish that *the bilinear form B is elliptic over the space $\mathbf{V}(\omega)$ if assumption* (2.10) *holds.* Note that, since this bilinear form is clearly continuous with respect to the norm $\|.\|_{\mathbf{H}^1(\omega)\times H^2(\omega)}$, *the existence and uniqueness of the solution of the variational problem* (2.2) *asserted in the next theorem follow from the Lax-Milgram lemma.*

Theorem 2.1 *: Assume that*

$$
(2.10) \qquad \textit{length } \gamma_0 > 0,
$$

and let the space $\mathbf{V}(\omega)$ be defined as in (2.3). Then the bilinear form B of (2.4) is $\mathbf{V}(\omega)$-elliptic, in the sense that there exists a constant β such that

$$
(2.11) \qquad \beta > 0 \textit{ and } B(\boldsymbol{\eta}, \boldsymbol{\eta}) \geq \beta\|\boldsymbol{\eta}\|^2_{\mathbf{H}^1(\omega)\times H^2(\omega)} \textit{ for all } \boldsymbol{\eta} \in \mathbf{V}(\omega),
$$

where the norm $\|.\|_{\mathbf{H}^1(\omega)\times H^2(\omega)}$ *is defined in (2.9).*

Consequently, if the linear form L is continuous with respect to the norm $\|.\|_{\mathbf{H}^1(\omega)\times H^2(\omega)}$, *Koiter's shell model (2.2) has one and only one solution.* ■

For convenience, the proof of Theorem 2.1 is arranged as a series of seven lemmas (Lemma 2.1 to Lemma 2.7).

Lemma 2.1 : *There exists a constant* C_1 *such that*

$$(2.12) \qquad C_1 > 0 \text{ and } B(\boldsymbol{\eta},\boldsymbol{\eta}) \geq C_1 \left\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 + \sum_{\alpha,\beta} \|\Upsilon_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 \right\}$$

for all $\boldsymbol{\eta} \in \mathbf{H}^1(\omega) \times H^2(\omega)$ *(the functions* $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ *and* $\Upsilon_{\alpha\beta}(\boldsymbol{\eta})$ *are defined in (2.6) and (2.7)).*

Proof : Let $(T_{\alpha\beta})$ denotes an arbitrary symmetric tensor. On the one hand,

$$(2.13) \qquad a^{\alpha\beta}a^{\rho\sigma}T_{\rho\sigma}T_{\alpha\beta} = (a^{\alpha\beta}T_{\alpha\beta})^2 \geq 0.$$

On the other, there exists a constant $C > 0$ such that

$$(2.14) \qquad a^{\alpha\rho}(x^1,x^2)a^{\beta\sigma}(x^1,x^2)T_{\rho\sigma}T_{\alpha\beta} \geq CT_{\alpha\beta}T_{\alpha\beta}$$

at all points $(x^1,x^2) \in \bar{\omega}$. To see this, we observe that the left-hand side of the last inequality may be written as $\boldsymbol{\theta}\,{}^t\mathbf{A}\boldsymbol{\theta}$, with

$$\mathbf{A} = \mathbf{A}^t = \begin{pmatrix} a^{11}a^{11} & 2a^{11}a^{12} & a^{12}a^{12} \\ 2a^{11}a^{12} & 2(a^{12}a^{12}+a^{11}a^{22}) & 2a^{12}a^{22} \\ a^{12}a^{12} & 2a^{12}a^{22} & a^{22}a^{22} \end{pmatrix}, \quad \boldsymbol{\theta} = \begin{pmatrix} T_{11} \\ T_{12} \\ T_{22} \end{pmatrix}.$$

Since $a^{11}a^{11} > 0$,

$$\det \begin{pmatrix} a^{11}a^{11} & 2a^{11}a^{12} \\ 2a^{11}a^{12} & 2(a^{12}a^{12}+a^{11}a^{22}) \end{pmatrix} = 2\frac{a^{11}a^{11}}{a} > 0,$$

and since $\det \mathbf{A} = 2/a^3 > 0$ (cf. (1.9)), we infer from a well-known characterization that the symmetric matrix $\mathbf{A}$ is positive definite at all points $(x_1,x_2) \in \bar{\omega}$. Hence there exists such a constant C.

Inequality (2.12) is then an immediate consequence of the specific form of the integrand that appears in the bilinear form $B(.,.)$, of the assumed positiveness of the Lamé constants (cf. (2.1)), and of inequalities (2.13) and (2.14). ■

Remark : A proof of Lemma 2.1 was already given by Rougée [1969, Chapter 2]. ■

Lemma 2.2 : *Let the space* $\mathbf{E}(\omega)$ *be defined by*

$$(2.15) \quad \mathbf{E}(\omega) = \{\boldsymbol{\eta} = ((\eta_\alpha),\eta_3) \in \mathbf{L}^2(\omega) \times H^1(\omega)\ ;\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega),\ \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)\},$$

where the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ *and* $\Upsilon_{\alpha\beta}(\boldsymbol{\eta})$ *are defined as in (2.6) and (2.7). Then*

$$(2.16) \qquad \mathbf{E}(\omega) = \mathbf{H}^1(\omega) \times H^2(\omega).$$

Proof : In definition (2.15), the relations "$\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$" and "$\Upsilon_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$" are to be understood in the sense of distributions ; this means that a function $\boldsymbol{\eta} = ((\eta_\alpha), \eta_3) \in \mathbf{L}^2(\omega) \times H^1(\omega)$ belongs to the space $\mathbf{E}(\omega)$ if and only if there exist functions $\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ and $\Upsilon_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ such that

$$(2.17) \quad \int_\omega \gamma_{\alpha\beta}(\boldsymbol{\eta})\psi dx^1 dx^2 = -\int_\omega \left\{\frac{1}{2}(\eta_\beta \partial_\alpha \psi + \eta_\alpha \partial_\beta \psi) + \Gamma^\rho_{\alpha\beta}\eta_\rho \psi + b_{\alpha\beta}\eta_3 \psi\right\} dx^1 dx^2,$$

and

$$(2.18) \quad \left\{ \begin{aligned} &\int_\omega \Upsilon_{\alpha\beta}(\boldsymbol{\eta})\psi dx^1 dx^2 = -\int_\omega \left\{\partial_\alpha \eta_3 \partial_\beta \psi + \Gamma^\rho_{\alpha\beta}\partial_\rho \eta_3 \psi + b^\rho_\beta(\eta_\rho \partial_\alpha \psi + \Gamma^\sigma_{\rho\alpha}\eta_\sigma \psi)\right. \\ &\qquad \left. + b^\rho_\alpha(\eta_\rho \partial_\beta \psi + \Gamma^\sigma_{\rho\beta}\eta_\sigma \psi) - (\partial_\alpha b^\rho_\beta + \Gamma^\rho_{\alpha\sigma} b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta} b^\rho_\sigma)\eta_\rho \psi + c_{\alpha\beta}\eta_3 \psi\right\} dx^1 dx^2 \end{aligned} \right.$$

for all functions $\psi \in \mathcal{D}(\omega)$. Note that the assumption $\boldsymbol{\varphi} \in \mathcal{C}^3(\bar{\omega})$ is used here, to insure that the functions $\partial_\alpha b^\rho_\beta$ are continuous on $\bar{\omega}$. It *a fortiori* implies that all other factors appearing in eqs. (2.17)-(2.18) (such as $\Gamma^\rho_{\alpha\beta}$, $b_{\alpha\beta}$, etc.) are also continuous on $\bar{\omega}$.

Let $\boldsymbol{\eta}$ be an arbitrary element in the space $\mathbf{E}(\omega)$. The relations (cf. (2.6)) :

$$e_{\alpha\beta}(\boldsymbol{\eta}) = \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \Gamma^\rho_{\alpha\beta}\eta_\rho + b_{\alpha\beta}\eta_3$$

then show that $e_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$. Therefore the identities (in the sense of distributions)

$$\partial_{\alpha\beta}\eta_\rho = \partial_\alpha e_{\beta\rho}(\boldsymbol{\eta}) + \partial_\beta e_{\alpha\rho}(\boldsymbol{\eta}) - \partial_\rho e_{\alpha\beta}(\boldsymbol{\eta})$$

show that the distributions $\partial_{\alpha\beta}\eta_\rho = \partial_\beta(\partial_\alpha \eta_\rho)$ belong to the space $H^{-1}(\omega)$. Since $\eta_\rho \in L^2(\omega)$ implies $\partial_\alpha \eta_\rho \in H^{-1}(\omega)$, a *lemma of J.L. Lions* (mentioned for the first time in Magenes & Stampacchia [1958], and found also in Duvaut & Lions [1972, p. 110], and Borchers & Sohr [1990] or Amrouche [1990], Amrouche & Girault [1990] for the extension to domains with Lipschitz-continuous boundaries) shows that the distributions $\partial_\alpha \eta_\rho$ are in fact in $L^2(\omega)$. Therefore

$$\eta_\alpha \in H^1(\omega),$$

and it follows form the definition (2.7) of the functions $\Upsilon_{\alpha\beta}(\boldsymbol{\eta})$ that the functions $\partial_{\alpha\beta}\eta_3$ belongs to the space $L^2(\omega)$. Hence

$$\eta_3 \in H^2(\omega),$$

and thus the space $\mathbf{E}(\omega)$ is contained in the space $\mathbf{H}^1(\omega) \times H^2(\omega)$. Since the space $\mathbf{H}^1(\omega) \times H^2(\omega)$ is clearly in the space $\mathbf{E}(\omega)$, the conclusion follows. ■

Lemma 2.3 : *There exists a constant* C_2 *such that*

$$(2.19) \qquad C_2 > 0 \text{ and } \|\boldsymbol{\eta}\| \geq C_2 \|\boldsymbol{\eta}\|_{\mathbf{H}^1(\omega)\times H^2(\omega)} \text{ for all } \boldsymbol{\eta} \in \mathbf{H}^1(\omega) \times H^2(\omega),$$

where

$$(2.20) \qquad \|\boldsymbol{\eta}\| := \left\{\sum_\alpha \|\eta_\alpha\|^2_{0,\omega} + \|\eta_3\|^2_{1,\omega} + \sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} + \sum_{\alpha,\beta}\|\Upsilon_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega}\right\}^{1/2}.$$

Hence $\|.\|$ and $\|.\|_{\mathbf{H}^1(\omega)\times H^2(\omega)}$ are equivalent norms over the space $\mathbf{H}^1(\omega)\times H^2(\omega)$ (the other inequality clearly holds).

Proof : When equipped with the norm (2.20), the space $\mathbf{E}(\omega)$ becomes a Hilbert space (to see this, consider a Cauchy sequence and for a fixed function $\psi \in \mathcal{D}(\omega)$, pass to the limit in equations (2.17) and (2.18)). Since the identity mapping from the space $\mathbf{H}^1(\omega)\times H^2(\omega)$ into the space $\mathbf{E}(\omega)$ is continuous (this follows from the definitions (2.9) and (2.20) of the norms $\|.\|_{\mathbf{H}^1(\omega)\times H^2(\omega)}$ and $\|.\|$) and onto by Lemma 2.2, and since both spaces are complete, the conclusion follows by the open mapping theorem. ■

To complete the proof of Theorem 2.1, it remains to show that, when the boundary conditions $\eta_i = \partial_\nu \eta_3 = 0$ on γ_0 are taken into account, inequality (2.19) (possibly with another constant) remains valid if in its left-hand side, the *norm* $\|\boldsymbol{\eta}\|$ of (2.20) is replaced by the *semi-norm* $\left\{\sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} + \sum_{\alpha,\beta}\|\Upsilon_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega}\right\}^{1/2}$. In other words, we need to show that this semi-norm is in fact a norm over the space $\mathbf{V}(\omega)$ of (2.3)). The proof of this fact will require four additional lemmas (Lemmas 2.4 to 2.7) ; note that Lemmas 2.4 to 2.6 were essentially contained in Bernadou & Ciarlet [1976, Theorems 5.1.1 and 5.2.1]. The presentation given here is slightly different, however.

Lemma 2.4 : *With an arbitrary element $\boldsymbol{\eta} = (\eta_i)$ in the space $\mathbf{H}^1(\omega)\times H^2(\omega)$ (i.e., $\eta_\alpha \in H^1(\omega)$ and $\eta_3 \in H^2(\omega)$), we associate the functions (cf. (1.6) and (1.16) for the definitions of $\varepsilon_{\alpha\beta}$ and $\eta_{\beta|\alpha}$) :*

$$d^\alpha(\boldsymbol{\eta}) = \varepsilon^{\alpha\beta}(\partial_\beta \eta_3 + b^\rho_\beta \eta_\rho) \in H^1(\omega), \quad d^3(\boldsymbol{\eta}) = \frac{1}{2}\,\varepsilon^{\alpha\beta}\eta_{\beta|\alpha} \in L^2(\omega). \tag{2.21}$$

Then the following relations hold (as equalities in the space $H^{-1}(\omega)$) :

$$\partial_\alpha[d^i(\boldsymbol{\eta})\mathbf{a}_i] = \varepsilon^{\rho\beta}[\Upsilon_{\alpha\beta}(\boldsymbol{\eta}) - b^\sigma_\alpha \gamma_{\sigma\beta}(\boldsymbol{\eta})]\mathbf{a}_\rho + \varepsilon^{\rho\beta}[\gamma_{\alpha\beta}(\boldsymbol{\eta})|_\rho]\mathbf{a}_3, \tag{2.22}$$

$$\partial_\alpha[\eta_i \mathbf{a}^i - (d^i(\boldsymbol{\eta})\mathbf{a}_i)] \times \boldsymbol{\varphi} = \gamma_{\alpha\beta}(\boldsymbol{\eta})\mathbf{a}^\beta - [\partial_\alpha(d^i(\boldsymbol{\eta})\mathbf{a}_i)] \times \boldsymbol{\varphi}, \tag{2.23}$$

where the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\Upsilon_{\alpha\beta}(\boldsymbol{\eta})$ are defined as in (2.6)-(2.7) (cf. (1.17) for the definition of $\gamma_{\alpha\beta}(\boldsymbol{\eta})|_\rho$), and $\boldsymbol{\varphi}$ is the mapping that defines the surface $S = \boldsymbol{\varphi}(\bar{\omega})$.

Proof : We first mention that the particular forms (2.21) of the functions $d^i(\boldsymbol{\eta})$ can be *a priori* justified (cf. the Remark that follows the proof of Lemma 2.5). For the sake of conciseness we let

$$d^i = d^i(\boldsymbol{\eta}), \quad \gamma_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\eta}), \quad \Upsilon_{\alpha\beta} = \Upsilon_{\alpha\beta}(\boldsymbol{\eta})$$

throughout the proof, and we leave it to the reader to verify that, at each stage, all the computations are valid in the distributional sense ; the assumption that $\boldsymbol{\varphi} : \bar{\omega} \to \mathcal{R}^3$ is of class $\mathcal{C}^3$ is in particular needed for that purpose.

Using the definition of the functions d^α, relations (1.18) and (1.29), and the definition of the functions $\Upsilon_{\alpha\beta}$, we obtain

$$d^\rho|_\alpha = \varepsilon^{\rho\beta}(\eta_{3|\alpha\beta} + b^\sigma_\beta|_\alpha \eta_\sigma + b^\sigma_\beta \eta_{\sigma|\alpha}) = \varepsilon^{\rho\beta}(\Upsilon_{\alpha\beta} - b^\sigma_\alpha \eta_{\sigma|\beta} + c_{\alpha\beta}\eta_3), \tag{2.24}$$

and, using the definition of the function d^3 and formula (1.25), we verify that

$$b_\alpha^\rho d^3 = \frac{1}{2}\, b_\alpha^\sigma \varepsilon^{\rho\beta}(\eta_{\beta|\sigma} - \eta_{\sigma|\beta}). \tag{2.25}$$

Using the definition (1.14) of the third fundamental form and the definition of the functions $\gamma_{\alpha\beta}$, we next have

$$b_\alpha^\sigma \eta_{\sigma|\beta} = b_\alpha^\sigma \gamma_{\sigma\beta} - \frac{1}{2}\, b_\alpha^\sigma(\eta_{\beta|\sigma} - \eta_{\sigma|\beta}) + c_{\alpha\beta}\eta_3, \tag{2.26}$$

and thus, (2.24), (2.25), and (2.26) yield

$$d^\rho|_\alpha = \varepsilon^{\rho\beta}(\Upsilon_{\alpha\beta} - b_\alpha^\sigma\gamma_{\sigma\beta}) + b_\alpha^\rho d^3. \tag{2.27}$$

Using the definition of the function d^3 and relation (1.18), we obtain

$$d^3|_\alpha = \frac{1}{2}\, \varepsilon^{\rho\beta}(\eta_{\rho|\beta})|_\alpha = \frac{1}{2\sqrt{a}}\, [(\eta_{2|1})|_\alpha - (\eta_{1|2})|_\alpha], \tag{2.28}$$

and, using the definitions of the functions d^α and $\varepsilon^{\alpha\beta}$ and relation (1.27), we have

$$\left\{\begin{aligned} b_{\alpha\beta}d^\beta &= \frac{1}{\sqrt{a}}\, [b_{1\alpha}\partial_2\eta_3 - b_{2\alpha}\partial_1\eta_3 + (b_{1\alpha}b_2^\rho - b_{2\alpha}b_1^\rho)\eta_\rho] \\ &= \frac{1}{\sqrt{a}}\, [b_{1\alpha}\partial_2\eta_3 - b_{2\alpha}\partial_1\eta_3 + (\eta_{\alpha|2})|_1 - (\eta_{\alpha|1})|_2]. \end{aligned}\right. \tag{2.29}$$

The Mainardi-Codazzi identities (1.21) and the definition of the functions $\gamma_{\alpha\beta}$ next yield

$$\left\{\begin{aligned} \varepsilon^{\rho\beta}\gamma_{\alpha\beta}|_\rho &= \frac{1}{\sqrt{a}}\left[\frac{1}{2}\,(\eta_{\alpha|2})|_1 + \frac{1}{2}\,(\eta_{2|\alpha})|_1 - (b_{\alpha2|1})\eta_3 - b_{\alpha2}\partial_1\eta_3\right] \\ &\quad + \frac{1}{\sqrt{a}}\left[-\frac{1}{2}\,(\eta_{\alpha|1})|_2 - \frac{1}{2}\,(\eta_{1|\alpha})|_2 + (b_{\alpha1|2})\eta_3 + b_{\alpha1}\partial_2\eta_3\right] \\ &= \frac{1}{2\sqrt{a}}\, [(\eta_{\alpha|2})|_1 + (\eta_{2|\alpha})|_1 - (\eta_{\alpha|1})|_2 - (\eta_{1|\alpha})|_2] \\ &\quad + \frac{1}{\sqrt{a}}\, [b_{\alpha1}\partial_2\eta_3 - b_{\alpha2}\partial_1\eta_3]; \end{aligned}\right. \tag{2.30}$$

and thus, (2.28), (2.29), and (2.30) together imply that

$$d^3|_\alpha + b_{\alpha\beta}d^\beta - \varepsilon^{\rho\beta}\gamma_{\alpha\beta|\rho} = 0. \tag{2.31}$$

Since, by (1.22),

$$\partial_\alpha(d^i\mathbf{a}_i) = (d^\rho|_\alpha - b_\alpha^\rho d^3)\mathbf{a}_\rho + (d^3|_\alpha + b_{\alpha\beta}d^\beta)\mathbf{a}_3, \tag{2.32}$$

equality (2.22) follows form (2.27), (2.31) and (2.32).

Using relations (1.5) and (1.6) and the definitions of the functions d^i, we next obtain

$$[d^i\mathbf{a}_i] \times \mathbf{a}_\alpha = \frac{1}{2}\, \varepsilon_{\alpha\rho}\varepsilon^{\sigma\beta}\eta_{\beta|\sigma}\mathbf{a}^\rho + (\partial_\alpha\eta_3 + b_\alpha^\rho\eta_\rho)\mathbf{a}^3. \tag{2.33}$$

But, using definition of the functions $\gamma_{\alpha\beta}$, we can also write

$$\frac{1}{2}\,\varepsilon_{\alpha\rho}\varepsilon^{\sigma\beta}\eta_{\beta|\sigma}\mathbf{a}^{\rho} = (\eta_{\rho|\alpha} - b_{\rho\alpha}\eta_3)\mathbf{a}^{\rho} - \gamma_{\rho\alpha}\mathbf{a}^{\rho}\;; \tag{2.34}$$

hence we conclude from (2.33) and (2.34) that

$$(d^i\mathbf{a}_i) \times \mathbf{a}_\alpha = (\eta_{\rho|\alpha} - b_{\rho\alpha}\eta_3)\mathbf{a}^{\rho} + (\partial_\alpha\eta_3 + b^{\rho}_{\alpha}\eta_\rho)\mathbf{a}^3 - \gamma_{\rho\alpha}\mathbf{a}^{\rho}, \tag{2.35}$$

on the one hand. The formula (1.22) shows that

$$\partial_\alpha[\eta_i\mathbf{a}^i] = (\eta_{\rho|\alpha} - b_{\rho\alpha}\eta_3)\mathbf{a}^{\rho} + (\eta_{3|\alpha} + b^{\rho}_{\alpha}\eta_\rho)\mathbf{a}^3 \tag{2.36}$$

on the other hand. Hence (2.35) and (2.36) imply that

$$(d^i\mathbf{a}_i) \times \mathbf{a}_\alpha = \partial_\alpha(\eta_i\mathbf{a}^i) - \gamma_{\alpha\rho}\mathbf{a}^{\rho}. \tag{2.37}$$

Since $\mathbf{a}_\alpha = \partial_\alpha\boldsymbol{\varphi}$ (cf. (1.1)), we also have

$$(d^i\mathbf{a}_i) \times \mathbf{a}_\alpha = \partial_\alpha[(d^i\mathbf{a}_i) \times \boldsymbol{\varphi}] - [\partial_\alpha(d^i\mathbf{a}_i)] \times \boldsymbol{\varphi}, \tag{2.38}$$

and equality (2.23) follows from (2.37) and (2.38). ■

We are now in a position to prove a key *infinitesimal rigid displacement lemma*, which plays in linearly elastic shell theory the same rôle as that played by the well known infinitesimal rigid displacement lemma in three-dimensional linearized elasticity (cf. e.g. Ciarlet [1988, p. 295.]).

Lemma 2.5 : *Let $\boldsymbol{\eta} = (\eta_i)$ be an element in the space $\mathbf{H}^1(\omega) \times H^2(\omega)$ that satisfies the following relations :*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) = 0, \tag{2.39}$$

as equalities in $L^2(\omega)$. Then there exist two vectors $\mathbf{c} \in \mathcal{R}^3$ and $\mathbf{d} \in \mathcal{R}^3$ such that

$$\eta_i(x^1,x^2)\mathbf{a}^i(x^1,x^2) = \mathbf{c} + \mathbf{d} \times \boldsymbol{\varphi}(x^1,x^2) \textit{ for all } (x^1,x^2) \in \bar{\omega}. \tag{2.40}$$

Furthermore, the functions $\eta_i : \bar{\omega} \to \mathcal{R}$ are of class C^2, and the constant vector $\mathbf{d}$ is given by

$$\mathbf{d} = \varepsilon^{\alpha\beta}(\partial_\beta\eta_3 + b^{\rho}_{\beta}\eta_\rho)\mathbf{a}_\alpha + \frac{1}{2}\,\varepsilon^{\alpha\beta}\eta_{\beta|\alpha}\mathbf{a}_3. \tag{2.41}$$

Proof : Since a distribution whose partial derivatives of the first order vanish on a connected open set is a constant function (see Schwartz [1966, p. 60]), and since the relations $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ imply $\gamma_{\alpha\beta}(\boldsymbol{\eta})|_\rho = 0$, we conclude from (2.22) that

$$\mathbf{d} := d^i(\boldsymbol{\eta})\mathbf{a}_i = \varepsilon^{\alpha\beta}(\partial_\beta\eta_3 + b^{\rho}_{\beta}\eta_\rho)\mathbf{a}_\alpha + \frac{1}{2}\,\varepsilon^{\alpha\beta}\eta_{\beta|\alpha}\mathbf{a}_3$$

is a *constant* vector, i.e., a vector of $\mathcal{R}^3$ that does *not* depend on $(x^1,x^2) \in \bar{\omega}$. We next conclude from (2.23) that

$$\mathbf{c} := \eta_i\mathbf{a}^i - \mathbf{d} \times \boldsymbol{\varphi}$$

is likewise a *constant* vector, and relation (2.40) is thus established. Finally, we infer from (1.4) and (2.40) that

$$\eta_j = \mathbf{c}.\mathbf{a}_j + (\mathbf{d} \times \boldsymbol{\varphi}).\mathbf{a}_j;$$

hence each function η_j is indeed of class $\mathcal{C}^2$. ■

Remark : We are now in a position to explain how the specific expressions (2.21) were chosen for the functions $d^i(\boldsymbol{\eta})$. Let a *priori*

$$\eta_i(x^1, x^2)\mathbf{a}_i(x^1, x^2) = \mathbf{c} + \mathbf{d} \times \boldsymbol{\varphi}(x^1, x^2) \text{ for all } (x^1, x^2) \in \bar{\omega},$$

where $\mathbf{c}$ and $\mathbf{d}$ are two constant vectors in $\mathcal{R}^3$. Letting $\mathbf{d} = d^i\mathbf{a}_i$, one easily verifies that

$$\eta_{\beta|\alpha} = b_{\alpha\beta}\eta_3 + \varepsilon_{\alpha\beta}d^3, \quad \eta_{3|\beta} = -b^\rho_\beta\eta_\rho + \varepsilon_{\rho\beta}d^\rho,$$

and then that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) = 0$. Using the relations $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$ and $\varepsilon^{\alpha\rho}\varepsilon_{\rho\beta} = -\delta^\alpha_\beta$, one next obtains

$$\varepsilon^{\alpha\beta}\eta_{\beta|\alpha} = \varepsilon^{\alpha\beta}b_{\beta\alpha}\eta_3 + \varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta}d^3 = 2d^3,$$

$$\varepsilon^{\alpha\beta}(\partial_\beta\eta_3 + b^\rho_\beta\eta_\rho) = \varepsilon^{\alpha\beta}\varepsilon_{\rho\beta}d^\rho = d^\alpha,$$

i.e., precisely the defining relations (2.21). ■

We are now able to take into account the imposed boundary conditions :

Lemma 2.6 *: Let $\boldsymbol{\eta} = (\eta_i)$ be an element in the space $\mathbf{H}^1(\omega) \times H^2(\omega)$ that satisfies relations (2.39) and the boundary conditions*

(2.42) $$\eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0 \subset \gamma, \text{ with length } \gamma_0 > 0.$$

Then $\boldsymbol{\eta} = \mathbf{0}$.

Proof : By Lemma 2.5, the relations $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) = 0$ imply that there exist two constant vectors $\mathbf{c}$ and $\mathbf{d}$ such that

$$\eta_i\mathbf{a}^i = \mathbf{c} + \mathbf{d} \times \boldsymbol{\varphi}.$$

The boundary conditions $\eta_i = \partial_\nu\eta_3 = 0$ on γ_0 imply that the constant vector $\mathbf{d}$ as given in (2.41) reduces to

(2.43) $$\mathbf{d} = \frac{1}{2}\,\varepsilon^{\alpha\beta}\eta_{\beta|\alpha}\mathbf{a}_3$$

along γ_0, since $\partial_\beta\eta_3 + b^\rho_\beta\eta_\rho = 0$ on γ_0 (observe that $\eta_{\beta|\alpha}$ is well defined along γ_0 since $\eta_\beta \in \mathcal{C}^2(\bar{\omega})$ by Lemma 2.5). Let x and x_0 be two distinct points on γ_0, so that $\boldsymbol{\varphi}(x) \neq \boldsymbol{\varphi}(x_0)$ (the mapping $\boldsymbol{\varphi} : \bar{\omega} \to \mathcal{R}^3$ is assumed to be injective). Since

$$\mathbf{c} + \mathbf{d} \times \boldsymbol{\varphi}(x) = \mathbf{c} + \mathbf{d} \times \boldsymbol{\varphi}(x_0) = \mathbf{0}$$

($\eta_i\mathbf{a}^i = \mathbf{0}$ on γ_0 by assumption), it follows that

$$\mathbf{d} \times [\boldsymbol{\varphi}(x) - \boldsymbol{\varphi}(x_0)] = 0 \text{ for all } x \in \gamma_0.$$

Therefore, if $\mathbf{d} \neq \mathbf{0}$, there exists a line Δ parallel to $\mathbf{d}$ such that

(2.44) $$\varphi(x) \in \Delta \text{ for all } x \in \gamma_0.$$

But if this were the case, the vector $\mathbf{d}$ should be both *normal* to the surface $S = \varphi(\omega)$ (by (2.43)) and *tangent* to S (by (2.44)) at all points of γ_0 ; this is impossible.

Hence $\mathbf{d} = \mathbf{0}$; then the boundary conditions $\eta_i = 0$ on γ_0 imply that $\mathbf{c} = \mathbf{0}$. ∎

Lemma 2.7 : *Assume that length $\gamma_0 > 0$. Then the semi-norm $|.|$ defined by*

(2.45) $$|\boldsymbol{\eta}| := \left\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\Upsilon_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} \right\}^{1/2}$$

is a norm over the space $\mathbf{V}(\omega)$ defined in (2.3), and there exists a constant C_3 such that

(2.46) $$C_3 > 0 \textit{ and } |\boldsymbol{\eta}| \geq C_3 \|\boldsymbol{\eta}\|_{\mathbf{H}^1(\omega)\times H^2(\omega)} \textit{ for all } \boldsymbol{\eta} \in \mathbf{V}(\omega).$$

Hence the norm $|.|$ is equivalent to the norm $\|.\|_{\mathbf{H}^1(\omega)\times H^2(\omega)}$ over the space $\mathbf{V}(\omega)$ (the other inequality clearly holds).

Proof : That $|.|$ is a norm over the space $\mathbf{V}(\omega)$ follows from Lemma 2.6. If inequality (2.46) is false, there exists a sequence $(\boldsymbol{\eta}^k)$ of elements in the space $\mathbf{V}(\omega)$ such that

(2.47) $$\|\boldsymbol{\eta}^k\|_{\mathbf{H}^1(\omega)\times H^2(\omega)} = 1 \text{ for all } k,$$

(2.48) $$|\boldsymbol{\eta}^k| \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

By (2.47) and the Rellich-Kondrašov theorem, there exists a subsequence $(\boldsymbol{\eta}^\ell)$ that converges in the space $\mathbf{L}^2(\omega) \times H^1(\omega)$. Since $(\boldsymbol{\eta}^\ell) \to 0$ as $\ell \to \infty$ by (2.48), we conclude that $(\boldsymbol{\eta}^\ell)$ is a Cauchy sequence with respect to the norm $\|.\|$ defined in (2.20). By Lemma 2.3, this subsequence converges in the space $\mathbf{V}(\omega)$.

Let $\boldsymbol{\eta}$ be its limit. By (2.48), it satisfies $|\boldsymbol{\eta}| = 0$, and thus $\boldsymbol{\eta} = \mathbf{0}$ by Lemma 2.6. But this contradicts (2.47), and the proof is complete. ∎

The proof of Theorem 2.1 follows by combining inequalities (2.12) and (2.46), established in Lemma 2.1 and Lemma 2.7 respectively.

Remark : In a related shell model considered and discussed in Sanders [1959], Budiansky & Sanders [1967] and Koiter [1966,1970], the change of curvature tensor $\Upsilon_{\alpha\beta}(\boldsymbol{\eta}))$ of (2.7) is replaced by the tensor $(\Upsilon'_{\alpha\beta}(\boldsymbol{\eta}))$, where

$$\Upsilon'_{\alpha\beta}(\boldsymbol{\eta}) := \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) - \frac{1}{2}\left[b^\rho_\alpha \gamma_{\rho\beta}(\boldsymbol{\eta}) + b^\sigma_\beta \gamma_{\alpha\sigma}(\boldsymbol{\eta})\right].$$

Since the equations $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \Upsilon_{\alpha\beta}(\boldsymbol{\eta}) = 0$ are equivalent to the equations $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \Upsilon'_{\alpha\beta}(\boldsymbol{\eta}) = 0$, the present analysis can be applied *verbatim* to this model. ∎

3 EXISTENCE THEORY FOR NAGHDI'S MODEL

We consider as in Sect. 2 a shell with middle surface S, thickness $2\varepsilon > 0$, and Lamé constants λ, μ satisfying inequalities (2.1).In *Naghdi's approach* (cf. Naghdi [1963,1972]), *constant shear deformations* are allowed across the thickness of the shell, in the sense that the displacement of a point with coordinate x_3 along the normal vector $\mathbf{a}^3$ is of the form $\zeta_i \mathbf{a}^i + x_3 r_\alpha \mathbf{a}^\alpha$, where ξ_i are as before the covariant components of the displacement of the points of the middle surface S and r_α are the (linearized) covariant components of the rotation of the unit normal vector $\mathbf{a}^3$. Hence the points situated on a line normal to S again remain on a line after the deformation has taken place ; however, this line is no longer normal to the deformed middle surface in general. In such a model, there are therefore *five unknowns* : the three functions $\zeta_i : \bar{\omega} \to \mathcal{R}$ as in Sect. 2 and, in addition, the two functions $r_\alpha : \bar{\omega} \to \mathcal{R}$; cf. Figure 3.1.

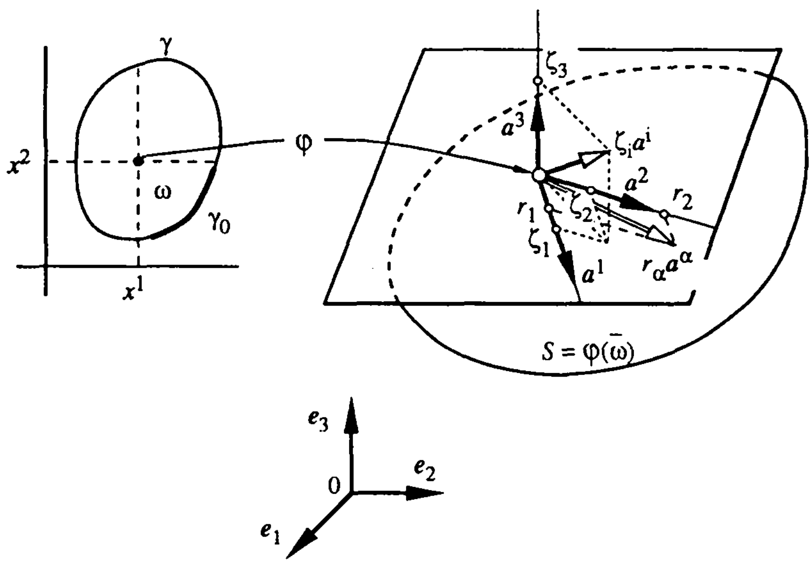

Figure 3.1 : In Naghdi's model, the five unknowns are the three covariant components $\zeta_i : \bar{\omega} \to \mathcal{R}$ of the displacement $\zeta_i \mathbf{a}^i$ of the points of the middle surface S of the shell and the two covariant components $x_\alpha : \bar{\omega} \to \mathcal{R}$ of the rotation $r_\alpha \mathbf{a}^\alpha$ of the unit normal vector $\mathbf{a}^3$.

Combining this geometrical assumption with the assumption of planar stress, P.M. Naghdi then obtains the following *two-dimensional problem* : *The unknown*

$$(\boldsymbol{\zeta}, \mathbf{r}) = ((\zeta_i), (r_\alpha))$$

solves the following variational problem, called **Naghdi's shell model** :

(3.1) $\qquad (\boldsymbol{\zeta}, \mathbf{r}) \in \check{\mathbf{V}}(\omega)$ and $\check{B}((\boldsymbol{\zeta}, \mathbf{r}), (\boldsymbol{\eta}, \mathbf{s})) = \check{L}(\boldsymbol{\eta}, \mathbf{s})$ for all $(\boldsymbol{\eta}, \mathbf{s}) \in \check{\mathbf{V}}(\omega)$,

where the space $\check{\mathbf{V}}(\omega)$ is defined as (here, $\mathbf{H}^1(\omega)$ denotes $[H^1(\omega)]^5$) :

(3.2) $\qquad \check{\mathbf{V}}(\omega) = \{(\boldsymbol{\eta}, \mathbf{s}) = ((\eta_i), (s_\alpha)) \in \mathbf{H}^1(\omega) \; ; \; \eta_i = s_\alpha = 0 \text{ on } \gamma_0\}$,

where the symmetric bilinear form $\check{B}$ is defined by

$$
(3.3) \quad \begin{cases} \tilde{B}((\boldsymbol{\zeta},\mathbf{r}),(\boldsymbol{\eta},\mathbf{s})) = \int_\omega \{\varepsilon a^{\alpha\beta\rho\sigma}\gamma_{\rho\sigma}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) \\ + \dfrac{\varepsilon^3}{3}\, a^{\alpha\beta\rho\sigma}\chi_{\rho\sigma}(\boldsymbol{\zeta},\mathbf{r})\chi_{\alpha\beta}(\boldsymbol{\eta},\mathbf{s}) + 8\varepsilon\mu a^{\alpha\beta}\gamma_{\alpha 3}(\boldsymbol{\zeta},\mathbf{r})\gamma_{\beta 3}(\boldsymbol{\eta},\mathbf{s})\}\sqrt{a}d\omega \end{cases}
$$

with (cf. (1.11)-(1.15) for the definitions of the functions $a^{\alpha\beta}$, $b_{\alpha\beta}$, b_α^β, $c_{\alpha\beta}$, $\Gamma_{\alpha\beta}^\rho$ and (1.16) for the definition of covariant derivatives) :

$$
(3.4) \quad a^{\alpha\beta\rho\sigma} = \frac{4\lambda\mu}{(\lambda+2\mu)}\, a^{\alpha\beta}a^{\rho\sigma} + 2\mu(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}),
$$

$$
(3.5) \quad \begin{cases} \gamma_{\alpha\beta}(\boldsymbol{\eta}) = \dfrac{1}{2}\,(\eta_{\alpha|\beta} + \eta_{\beta|\alpha}) - b_{\alpha\beta}\eta_3 \\ \quad = e_{\alpha\beta}(\boldsymbol{\eta}) - \Gamma_{\alpha\beta}^\rho\eta_\rho - b_{\alpha\beta}\eta_3, \text{ with } e_{\alpha\beta}(\boldsymbol{\eta}) = \tfrac{1}{2}\,(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha), \end{cases}
$$

$$
(3.6) \quad \begin{cases} \chi_{\alpha\beta}(\boldsymbol{\eta},\mathbf{s}) = \dfrac{1}{2}\,(s_{\alpha|\beta} + s_{\beta|\alpha}) - \dfrac{1}{2}\,b_\alpha^\rho(\eta_{\rho|\beta} - b_{\rho\beta}\eta_3) - \dfrac{1}{2}\,b_\beta^\sigma(\eta_{\sigma|\alpha} - b_{\sigma\alpha}\eta_3) \\ = e_{\alpha\beta}(\mathbf{s}) - \Gamma_{\alpha\beta}^\rho s_\rho - \dfrac{1}{2}\,b_\alpha^\rho(\partial_\beta\eta_\rho - \Gamma_{\rho\beta}^\sigma\eta_\sigma) - \dfrac{1}{2}\,b_\beta^\sigma(\partial_\alpha\eta_\sigma - \Gamma_{\alpha\sigma}^\rho\eta_\rho) + c_{\alpha\beta}\eta_3, \\ \quad \text{with } e_{\alpha\beta}(\mathbf{s}) = \dfrac{1}{2}\,(\partial_\alpha s_\beta + \partial_\beta s_\alpha), \end{cases}
$$

$$
(3.7) \quad \gamma_{\alpha 3}(\boldsymbol{\eta},\mathbf{s}) = \frac{1}{2}\,(\partial_\alpha\eta_3 + b_\alpha^\rho\eta_\rho + s_\alpha),
$$

and where $\tilde{L}$ is the linear form that takes into account the applied forces. The tensors $(a^{\alpha\beta\rho\sigma})$ and $(\gamma_{\alpha\beta}(\boldsymbol{\eta}))$ are the same (cf. (2.5) and (2.6)) as those appearing in Koiter's model. The tensor $(\chi_{\alpha\beta}(\boldsymbol{\eta},\mathbf{s}))$ is the *linearized change of curvature tensor* associated with an arbitrary displacement and rotation fields, of the middle surface S and of the normal vectors $\mathbf{a}_3$, respectively. The tensor $(\gamma_{\alpha 3}(\boldsymbol{\eta},s))$ is the *linearized transverse shear strain tensor.*

Remark : The linear form $\tilde{L}$ takes the form

$$
\tilde{L}(\boldsymbol{\eta},\mathbf{s}) = \int_\omega \mathbf{p}.\boldsymbol{\eta}\sqrt{a}d\omega + \int_{\gamma_1}(\mathbf{q}.\boldsymbol{\eta} + m^\alpha s_\alpha)d\gamma
$$

for all $(\boldsymbol{\eta},\mathbf{s}) \in \tilde{\mathbf{V}}(\omega)$, with $\gamma_1 = \gamma - \gamma_0$, where the vector fields $\mathbf{p} : \omega \to \mathcal{R}^3$, $\mathbf{q} : \gamma_1 \to \mathcal{R}^3$, and $(m^\alpha) : \gamma_1 \to \mathcal{R}^2$, are derived from the given applied body and surface forces acting on the shell, viewed as a three-dimensional body. ∎

Let

$$
(3.8) \quad \|(\boldsymbol{\eta},\mathbf{s})\|_{1,\omega} = \left\{\sum_i \|\eta_i\|_{1,\omega}^2 + \sum_\alpha \|s_\alpha\|_{1,\omega}^2\right\}^{1/2}
$$

denote the norm of an element $(\boldsymbol{\eta},\mathbf{s}) = ((\eta_i),(s_\alpha)) \in \tilde{\mathbf{V}}(\omega)$, where $\tilde{\mathbf{V}}(\omega)$ is the closed subspace of $\mathbf{H}^1(\omega)$ defined in (3.2). We now establish that the bilinear form $\tilde{B}$ is elliptic over the space $\tilde{V}(\omega)$ if assumption (3.9) holds.

Theorem 3.1 : *Assume that*

$$\text{(3.9)} \qquad \textit{length } \gamma_0 > 0,$$

and let the space $\tilde{\mathbf{V}}(\omega)$ *be defined as in (3.2). Then the bilinear form* $\tilde{B}$ *of (3.3) is* $\tilde{\mathbf{V}}(\omega)$*-elliptic, in the sense that there exists a constant* $\tilde{\beta}$ *such that*

$$\text{(3.10)} \qquad \tilde{\beta} > 0 \textit{ and } \tilde{B}((\boldsymbol{\eta}, \mathrm{s}), (\boldsymbol{\eta}, \mathrm{s})) \geq \tilde{\beta} \|(\boldsymbol{\eta}, \mathrm{s})\|_{1,\omega}^2 \textit{ for all } (\boldsymbol{\eta}, \mathrm{s}) \in \tilde{\underset{\sim}{V}}(\omega),$$

where the norm $\|.\|_{1,\omega}$ *is defined as in (3.8).*

Consequently, if the linear form $\tilde{L}$ *is continuous with respect to the norm* $\|.\|_{1,\omega}$*, Naghdi's shell model (3.1) has one and only one solution.* ■

The proof follows the same pattern as that of Theorem 2.1, and is accordingly arranged as a series of lemmas (Lemmas 3.1 to 3.6). Note however that the constants C_1, C_2, C_3 found below are not necessarily the same as those found in the lemmas used for proving Theorem 2.1 and that, likewise, the norms and semi-norms defined in (3.16) and (3.25) are not the same as those defined in (2.20) and (2.45).

Lemma 3.1 : *There exists a constant* C_1 *such that*

$$\text{(3.11)} \qquad \left\{ \begin{array}{c} C_1 > 0 \textit{ and } \tilde{B}((\boldsymbol{\eta}, \mathrm{s}), (\boldsymbol{\eta}, \mathrm{s})) \geq \\ C_1 \left\{ \sum\limits_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 + \sum\limits_{\alpha,\beta} \|\chi_{\alpha\beta}(\boldsymbol{\eta}, \mathrm{s})\|_{0,\omega}^2 + \sum\limits_{\alpha} \|\gamma_{\alpha 3}(\boldsymbol{\eta}, \mathrm{s})\|_{0,\omega}^2 \right\}^{1/2} \end{array} \right.$$

for all $(\boldsymbol{\eta}, \mathrm{s}) \in \mathbf{H}^1(\omega)$.

Proof : There exists a constant $C' > 0$ such that

$$\text{(3.12)} \qquad a^{\alpha\beta}(x^1, x^2) z_\alpha z_\beta \geq C' z_\alpha z_\alpha \text{ for all } (z_\alpha) \in \mathcal{R}^2$$

at all points $(x^1, x^2) \in \bar{\omega}$. The proof then follows by combining inequalities (2.13), (2.14), (3.12). ■

Lemma 3.2 : *Let the space* $\tilde{\mathbf{E}}(\omega)$ *be defined by (here* $\mathbf{L}^2(\omega) = [L^2(\omega)]^5$*)* :

$$\text{(3.13)} \qquad \left\{ \begin{array}{l} \tilde{\mathbf{E}}(\omega) = \{(\boldsymbol{\eta}, \mathrm{s}) \in \mathbf{L}^2(\omega) \quad ; \quad \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega), \quad \chi_{\alpha\beta}(\boldsymbol{\eta}, \mathrm{s}) \in L^2(\omega), \\ \qquad\qquad \gamma_{\alpha 3}(\boldsymbol{\eta}, \mathrm{s}) \in L^2(\omega)\}, \end{array} \right.$$

where the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$, $\chi_{\alpha\beta}(\boldsymbol{\eta}, \mathrm{s})$, $\gamma_{\alpha 3}(\boldsymbol{\eta}, \mathrm{s})$ *are defined as in (3.5), (3.6), (3.7). Then*

$$\text{(3.14)} \qquad \tilde{\mathbf{E}}(\omega) = \mathbf{H}^1(\omega).$$

Proof : Let $(\boldsymbol{\eta}, \mathrm{s})$ be an arbitrary element in the space $\tilde{\mathbf{E}}(\omega)$. First, the definition of $\gamma_{\alpha 3}(\boldsymbol{\eta}, \mathrm{s})$ implies that $\eta_3 \in H^1(\omega)$. The same argument, based on a *lemma of J.L. Lions*, as in the proof of Lemma 2.2, then shows that $\eta_\alpha \in H^1(\omega)$. Finally, the definition of $\chi_{\alpha\beta}(\boldsymbol{\eta}, \mathrm{s})$ implies that $e_{\alpha\beta}(\mathrm{s}) \in L^2(\omega)$, and the same lemma of J.L. Lions shows that $s_\alpha \in H^1(\omega)$. ■

Lemma 3.3 : *There exists a constant C_2 such that*

(3.15) $\quad C_2 > 0$ *and* $\|(\boldsymbol{\eta}, \mathrm{s})\| \geq C_2 \|(\boldsymbol{\eta}, \mathrm{s})\|_{1,\omega}$ *for all* $(\boldsymbol{\eta}, \mathrm{s}) = ((\eta_i), (s_\alpha)) \in \mathbf{H}^1(\omega)$,

where

$$
\left\{
\begin{aligned}
\|(\boldsymbol{\eta}, \mathrm{s})\| := \Bigg\{ & \sum_i \|\eta_i\|^2_{0,\omega} + \sum_\alpha \|s_\alpha\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} \\
& + \sum_{\alpha,\beta} \|\chi_{\alpha\beta}(\boldsymbol{\eta}, \mathrm{s})\|^2_{0,\omega} + \sum_\alpha \|\gamma_{\alpha 3}(\boldsymbol{\eta}, \mathrm{s})\|^2_{0,\omega} \Bigg\}^{1/2}.
\end{aligned}
\right. \tag{3.16}
$$

Hence $\|.\|$ *and* $\|.\|_{1,\omega}$ *are equivalent norms over the space* $\mathbf{H}^1(\omega)$ *(the other inequality clearly holds).*

Proof : The proof is analogous to that of Lemma 2.3, and for this reason, is omitted. ■

We now establish an *infinitesimal rigid displacement lemma*, which plays the same rôle for Naghdi's model as that played by Lemma 2.5 for Koiter's model. This result was first proved in Coutris [1978].

Lemma 3.4 : *Let* $(\boldsymbol{\eta}, \mathrm{s}) = ((\eta_i), (s_\alpha))$ *be an element in the space* $\mathbf{H}^1(\omega)$ *that satisfies*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \chi_{\alpha\beta}(\boldsymbol{\eta}, \mathrm{s}) = \gamma_{\alpha 3}(\boldsymbol{\eta}, \mathrm{s}) = 0, \tag{3.17}$$

as equalities in $L^2(\omega)$. Then there exist two vectors $\mathbf{c} \in \mathcal{R}^2$ and $\mathbf{d} \in \mathcal{R}^3$ such that

$$\eta_i(x^1, x^2)\mathbf{a}^i(x^1, x^2) = \mathbf{c} + \mathbf{d} \times \boldsymbol{\varphi}(x^1, x^2) \text{ for all } (x^1, x^2) \in \bar{\omega}. \tag{3.18}$$

Furthermore, the functions $\eta_i : \bar{\omega} \to \mathcal{R}$ are of class C^2, and the constant vector $\mathbf{d}$ is given by

$$\mathbf{d} = -\varepsilon^{\alpha\beta} s_\beta \mathbf{a}_\alpha + \frac{1}{2}\, \varepsilon^{\alpha\beta} \eta_{\beta|\alpha} \mathbf{a}^3. \tag{3.19}$$

Proof : From the relations $\gamma_{\alpha 3}(\boldsymbol{\eta}, \mathrm{s}) = 0$ and (1.19), (1.25), (1.29), (3.7), we infer that

$$2\gamma_{\alpha 3}(\boldsymbol{\eta}, \mathrm{s})|_\beta = \eta_{3|\alpha\beta} + (b^\rho_\alpha|_\beta)\eta_\rho + b^\rho_\alpha(\eta_{\rho|\beta}) + s_{\alpha|\beta} = 0, \tag{3.20}$$

or equivalently, that

$$
\left\{
\begin{aligned}
& \partial_{\alpha\beta}\eta_3 - \Gamma^\rho_{\alpha\beta}\partial_\rho\eta_3 + (\partial_\beta b^\rho_\alpha)\eta_\rho + \Gamma^\rho_{\beta\sigma} b^\sigma_\alpha \eta_\rho - \Gamma^\sigma_{\alpha\beta} b^\rho_\sigma \eta_\rho \\
& \quad + b^\rho_\alpha \partial_\beta \eta_\rho - b^\rho_\alpha \Gamma^\sigma_{\rho\beta}\eta_\sigma + \partial_\beta s_\alpha - \Gamma^\rho_{\alpha\beta} s_\rho = 0,
\end{aligned}
\right.
$$

which shows in particular that $\eta_3 \in H^2(\omega)$. Using (3.20) in definition (3.6), we next obtain

$$
\left\{
\begin{aligned}
-\chi_{\alpha\beta}(\boldsymbol{\eta}, \mathrm{s}) &= -\frac{1}{2}\,(s_{\alpha|\beta} + s_{\beta|\alpha}) + \frac{1}{2}\, b^\rho_\alpha(\eta_{\rho|\beta}) + \frac{1}{2}\, b^\sigma_\beta(\eta_{\sigma|\alpha}) - c_{\alpha\beta}\eta_3 \\
&= \eta_{3|\alpha\beta} + (b^\rho_\alpha|_\beta)\eta_\rho + b^\rho_\alpha(\eta_{\rho|\beta}) + b^\rho_\beta(\eta_{\rho|\alpha}) - c_{\alpha\beta}\eta_3
\end{aligned}
\right. \tag{3.21}
$$

(we need here the symmetry relations (1.20)). But since $\chi_{\alpha\beta}(\boldsymbol{\eta}, \mathbf{s}) = 0$ by assumption, and since the right-hand side of relation (3.21) is nothing but the definition (2.7) of the function $\Upsilon_{\alpha\beta}(\boldsymbol{\eta})$, we conclude that $\Upsilon_{\alpha\beta}((\boldsymbol{\eta}) = 0$.

Hence, by Lemma 2.5, there exist two constant vectors $\mathbf{c} \in \mathcal{R}^3$ and $\mathbf{d} \in \mathcal{R}^3$ such that relation (3.18) holds, the functions $\eta_i : \bar{\omega} \to \mathcal{R}$ are of class $\mathcal{C}^2$, and the constant vector $\mathbf{d}$ is given by (cf. (2.41))

$$\mathbf{d} = \varepsilon^{\alpha\beta}(\partial_\beta \eta_3 + b_\beta^\rho \eta_\rho)\mathbf{a}_\alpha + \frac{1}{2}\,\varepsilon^{\alpha\beta}\eta_{\beta|\alpha}\mathbf{a}_3 = -\varepsilon^{\alpha\beta}s_\beta \mathbf{a}_\alpha + \frac{1}{2}\,\varepsilon^{\alpha\beta}\eta_{\beta|\alpha}\mathbf{a}_3 \tag{3.22}$$

since $\gamma_{\beta 3}(\boldsymbol{\eta}, \mathbf{s}) = 0$. Relations (1.5) and (3.22) therefore imply that

$$\mathbf{d} \times \mathbf{a}_3 = \varepsilon^{\alpha\beta}\varepsilon_{\alpha\rho}s_\beta \mathbf{a}^\rho = s_\beta \mathbf{a}^\beta,$$

and thus the functions $s_\alpha : \bar{\omega} \to \mathcal{R}$ are also of class $\mathcal{C}^2$, since

$$s_\alpha = (\mathbf{d} \times \mathbf{a}_3).\mathbf{a}_\alpha.$$

■

We next consider the effect of the imposed boundary conditions :

Lemma 3.5 *: Let $(\boldsymbol{\eta}, \mathbf{s}) = ((\eta_i), (s_\alpha)$ be an element in the space $\mathbf{H}^1(\omega)$ that satisfies relations (3.17) and the boundary conditions*

$$\eta_i = s_\alpha = 0 \textit{ on } \gamma_0 \subset \gamma, \textit{ with length } \gamma_0 > 0. \tag{3.23}$$

Then $(\boldsymbol{\eta}, \mathbf{s}) = (\mathbf{0}, \mathbf{0})$.

Proof: By (3.22), the constant vector $\mathbf{d}$ satisfies

$$\mathbf{d} = \frac{1}{2}\,\varepsilon^{\alpha\beta}\eta_{\beta|\alpha}\mathbf{a}_3 \text{ on } \gamma_0 \tag{3.24}$$

(notice that $\eta_{\beta|\alpha}$ is well-defined on γ_0 since $\eta_\beta \in \mathcal{C}^2(\bar{\omega})$ by Lemma 3.4) since $s_\alpha = 0$ on γ_0. Since relation (3.24) coincides with relation (2.43), we may therefore continue the proof exactly as that of Lemma 2.6, since only the relations $\eta_i = 0$ on γ_0 are then needed. ■

Lemma 3.6 *: Assume that length $\gamma_0 > 0$. Then the semi-norm $|.|$ defined by*

$$|(\boldsymbol{\eta}, \mathbf{s})| := \left\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 + \sum_{\alpha,\beta} \|\chi_{\alpha\beta}(\boldsymbol{\eta}, \mathbf{s})\|_{0,\omega}^2 + \sum_{\alpha} \|\gamma_{\alpha 3}(\boldsymbol{\eta}, \mathbf{s})\|_{0,\omega}^2 \right\}^{1/2} \tag{3.25}$$

is a norm over the space $\tilde{\mathbf{V}}(\omega)$ defined in (3.2), and there exists a constant C_3 such that

$$C_3 > 0 \textit{ and } |(\boldsymbol{\eta}, \mathbf{s})| \geq C_3 \|(\boldsymbol{\eta}, \mathbf{s})\|_{1,\omega} \textit{ for all } (\boldsymbol{\eta}, \mathbf{s}) \in \tilde{\mathbf{V}}(\omega). \tag{3.26}$$

Hence the norm $|.|$ is equivalent to the norm $\|.\|_{1,\omega}$ over the space $\tilde{\mathbf{V}}(\omega)$ (the other inequality clearly holds).

Proof: The proof is analogous to that of Lemma 2.7 : That $|.|$ is a norm over the space $\tilde{\mathbf{V}}(\omega)$ follows from Lemma 3.5. If inequality (3.26) is false, there exists a sequence $((\boldsymbol{\eta}^k, \mathbf{s}^k))$ of elements in the space $\tilde{\mathbf{V}}(\omega)$ such that

$$\|((\boldsymbol{\eta}^k, \mathbf{s}^k))\|_{1,\omega} = 1 \text{ for all } k, \tag{3.27}$$

$$|((\boldsymbol{\eta}^k, \mathbf{s}^k))| \to 0 \text{ as } k \to \infty. \tag{3.28}$$

By (3.27) and the Rellich-Kondrašov theorem, there exists a subsequence $((\boldsymbol{\eta}^\ell, \mathbf{s}^\ell))$ that converges in the space $\mathbf{L}^2(\omega)$. Since $((\boldsymbol{\eta}^\ell, \mathbf{s}^\ell)) \to 0$ as $\ell \to \infty$ by (3.28), we conclude that defined in $((\boldsymbol{\eta}^\ell, \mathbf{s}^\ell))$ is a Cauchy sequence with respect to the norm $\|.\|$ defined in (3.16). By Lemma 3.3, this sequence converges in the space $\tilde{\mathbf{V}}(\omega)$.

Let $(\boldsymbol{\eta}, \mathbf{s})$ be its limit. By (3.28), it satisfies $|(\boldsymbol{\eta}, \mathbf{s})| = 0$, and thus $(\boldsymbol{\eta}, \mathbf{s}) = 0$ by Lemma 3.5. But this contradicts (3.27), and the proof is complete. ■

The proof of Theorem 3.2 follows by combining inequalities (3.11) and (3.26) established in Lemma 3.1 and 3.6 respectively.

Remark : Other shell models that also include constant shear deformations differ from Naghdi's model only by some strictly positive factor appearing in front of the "shear strain part" $\int_\omega a^{\alpha\beta}\gamma_{\alpha 3}(\boldsymbol{\zeta}, \mathbf{r})\gamma_{\beta 3}(\boldsymbol{\eta}, \mathbf{s})\sqrt{a}d\omega$ in the bilinear form (3.3). The present analysis clearly applies to such models. ■

REFERENCES

AMROUCHE, C. [1990] : Propriétés d'opérateurs de dérivation. Application au problème non homogène de Stokes, C.R. Acad. Sci. Paris, Série I, *310*, pp. 367-370.

AMROUCHE, C. ; GIRAULT, V. [1990] : Propriétés fonctionnelles d'opérateurs ; applications au problème de Stokes en dimension quelconque, Rapport R90025, Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, Paris.

BERNADOU, M. ; BOISSERIE, J.M. [1982] : The Finite Element Method in Thin Shell Theory : Application to Arch Dam Simulations, Birkhäuser, Boston.

BERNADOU, M. ; CIARLET, P.G. [1976] : Sur l'ellipticité du modèle linéaire de coques de W.T. Koiter. In *Computing Methods in Applied Sciences and Engineering* (R. Glowinski and J.L. Lions, Editors), Lectures Notes in Economics and Mathematical Systems, Vol. *134*, pp. 89-136, Springer-Verlag, Berlin.

BERNADOU, M. ; LALANNE, B. [1985] : Sur l'approximation des coques minces par des méthodes B-splines et éléments finis. In : *Tendances Actuelles en Calcul des Structures*, (J.P. Grellier and G.M. Campel, Editors), Editions Pluralis, Paris, pp. 939-958.

BORCHERS, W. ; SOHR, H. [1990] : On the equations *rot* $v = g$ and *div* $u = f$ with zero boundary conditions, Hokkaido Math. J., *19*, pp. 67-87.

BUDIANSKY, B. ; SANDERS, J.L. [1967] : On the "best" first-order linear shell theory, Progr. in Appl. Mech. (W. Prager Anniversary Volume), pp. 129-140, Macmillan, New York.

CIARLET, P.G. [1978] : *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam.

CIARLET, P.G. [1988] : *Mathematical Elasticity, Vol. I : Three-Dimensional Elasticity*, North-Holland, Amsterdam.

CIARLET, P.G. [1992a] : Echange de limites en théorie asymptotique de coques. I. En premier lieu, "la coque devient une plaque", C.R. Acad. Sci. Paris, Série I, *315*, pp. 107-111.

CIARLET, P.G. [1992b] : Echange de limites en théorie asymptotique de coques. II. En premier lieu, l'épaisseur tend vers zéro, C.R. Acad. Sci. Paris, Série I, *315*, pp. 227-233.

CIARLET, P.G. ; MIARA, B. [1991] : Une démonstration simple de l'ellipticité des modèles de coques de W.T. Koiter et de P.M. Naghdi, C.R. Acad. Sci. Paris, Série I, *312*, pp. 411-415.

CIARLET, P.G. ; MIARA, B. [1992a] : On the ellipticity of linear shell models, Z. angew Math. Phys., *43*, pp. 243-253.

CIARLET, P.G. ; MIARA, B. [1992b] : Justification of the two-dimensional equations of a linearly elastic shallow shell, Comm. Pure Appl. Math., *45*, pp. 327-360.

CIARLET, P.G. ; PAUMIER, J.C. [1986] : A justification of the Marguerre-von Kármán equations, Computational Mechanics, *1*, pp. 177-202.

COSSERAT, E. ; COSSERAT, F. [1909] : *Théorie des Corps Déformables*, Hermann, Paris.

COUTRIS, N. [1978] : Théorème d'existence et d'unicité pour un problème de coque élastique dans le cas d'un modèle linéaire de P.M. Naghdi, RAIRO Analyse Numérique, *12*, pp. 51-57.

DESTUYNDER, P. [1980] : *Sur une Justification des Modèles de Plaques et de Coques par les Méthodes Asymptotiques*, Doctoral Dissertation, Université Pierre et Marie Curie, Paris.

DESTUYNDER, P. [1985] : A classification of thin shell theories, Acta Applicandae Mathematicae, *4*, pp. 15-63.

DUVAUT, G. ; LIONS, J.L. [1972] : *Les Inéquations en Mécanique et en Physique*, Dunod, Paris.

GEYMONAT, G. ; SANCHEZ-PALENCIA, E. [1991] : Remarques sur la rigidité infinitésimale de certaines surfaces elliptiques non régulières, non convexes et applications, C.R. Acad. Sci. Paris, Série I, *313*, pp. 645-651.

GORDEZIANI, D.G. [1974] : On the solvability of some boundary value problems for a variant of the theory of thin shells, Dokl. Akad. Nauk SSSR, *215* (English translation : Soviet. Math. Dokl., *15* (1974), pp. 677-680).

GREEN, A.E. ; ZERNA, W. [1968] : *Theoretical Elasticity*, Oxford University Press, 2nd Edition.

JOHN, F. [1965] : Estimates for the derivatives of the stresses in a thin shell and interior shell equations, Comm. Pure Appl. Math., *18*, pp. 235-267.

JOHN, F. [1971] : Refined interior equations for thin elastic shells, Comm. Pure Appl. Math., *24*, pp. 583-615.

KIRCHHOFF, G. [1876] : Vorlesungen über Mathematische Physik, Mechanik, Leipzig.

KOITER, W.T. [1966] : On the nonlinear theory of thin elastic shells, Proc. Kon. Ned. Akad. Wetensch., *B69*, pp. 1-54.

KOITER, W.T. [1970] : On the foundations of the linear theory of thin elastic shells, Proc. Kon. Ned. Akad. Wetensch., *B73*, pp. 169-195.

LOVE, A.E.H. [1934] : *The Mathematical Theory of Elasticity*, Cambridge University Press.

MAGENES, E. ; STAMPACCHIA, G. [1958] : I problemi al contorno per le equazioni differenziali di tipo ellitico, Ann. Scuola Norm. Sup. Pisa, 12, pp. 247-358.

NAGHDI, P.M. [1963] : Foundations of elastic shell theory. In : *Progress in Solid Mechanics*, Vol. 4, pp. 1-90, North-Holland, Amsterdam.

NAGHDI, P.M. [1972] : The theory of shells and plates. In : *Handbuch der Physik*, Vol. VI a-2, pp. 425-640, Springer-Verlag, Berlin.

NEČAS, J. [1967] : *Les Méthodes Directes en Théorie des Equations Elliptiques*, Masson, Paris.

NOVOZHILOV, V.V. [1970] : *Thin Shell Theory*, Wolters-Noordhoff Publishing, Groningen.

PIILA, J. ; PITKÄRANTA, J. [1992] : Characterization of the membrane theory of a thin clamped shell of revolution. The parabolic case, Research Report A308, Helsinki University of Technology.

ROUGÉE, P. [1969] : *Equilibre des Coques Elastiques Minces Inhomogènes en Théorie non Linéaire*, Doctoral dissertation, Université de Paris.

SANCHEZ-PALENCIA, E. [1989a] : Statique et dynamique des coques minces. I. Cas de flexion pure non inhibée, C.R. Acad. Sci. Paris, Série I, *309*, pp. 411-417.

SANCHEZ-PALENCIA, E. [1989b] : Statique et dynamique des coques minces. II. Cas de flexion pure inhibée. Approximation membranaire, C.R. Acad. Sci. Paris, Série I, *309*, pp. 531-537.

SANCHEZ-PALENCIA, E. [1990] : Passage à la limite de l'élasticité tridimensionnelle à la théorie asymptotique des coques minces, C.R. Acad. Sci. Paris, Série II, ***311***, pp. 909-916.

SANDERS, J.L. [1959] : An improved first-approximation theory for thin shells, NASA Report 24.

SCHWARTZ, L. [1966] : *Théorie des Distributions*, Hermann, Paris.

SHOIKHET, B.A. [1974] : On existence theorems in linear shell theory, *PMM 38*, pp. 567-571 (English translation : Journal of Applied Mathematics and Mechanics, *38* (1974), pp. 527-531).

VALIRON, G. [1950] : *Equations fonctionnelles, Applications*, Masson, Paris (Second Edition).

VEKUA, I.N. [1965] : Theory of thin shallow shells of variable thickness, Akad. Nauk. Gruzin. SSR Trudy Tbilissi Mat. Inst. Razmadze, *30*, pp. 3-103 (Russian).

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